Math 210B Lecture 20 Notes

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February 27, 2019

1 Jordan Canonical Form

1.1 Existence and description of the Jordan canonical form

Let F be a field. Recall that an F-vector space V with a linear transformation $T : V \to V$ is the same as an $F[x]$-module V; The isomorphisms are

$$(V, T) \mapsto f(x) \cdot v = f(T)(v)$$

$$(V, x : V \to V) \leftarrow\!\shortmid V$$

This induces a correspondence between finite dimensional vector spaces with $T : V \to V$ and finitely generated torsion $F[x]$-modules V. A finitely generated torsion $F[x]$-module is

$$V \cong \bigoplus_{i=1}^{r} F[x]/(f_i)$$

where $f_i \in F[x]$ is monic with $\deg(f_i) = n_i$ and $f_1 \mid f_2 \mid \cdots \mid f_r$. Take the basis of V:

$$\{1, x, \dots, x^{n_1-1}, 1, x, \dots, x^{n_2-1}, \dots, 1, x, \dots, x^{n_r-1}\}$$

A matrix representing $x : V \to V$ with respect to this basis is

$$A = \begin{bmatrix} A_{f_1} & & & \\ & A_{f_2} & & \\ & & \ddots & \\ & & & A_{f_r} \end{bmatrix}.$$

$V_f = F[x]/(f)$, where f is monic, irreducible and of degree n has basis $1, x, \dots, x^{n-1}$. The matrix A_f representing $x : V_f \to V_f$ is determined by:

$$x \cdot x^{i-1} = x^i, \qquad 1 \leq i \leq n-1$$

$$x \cdot x^{n-1} = x^n = -\sum_{i=1}^{n-1} c_i x^i,$$

where $f = \sum_{i=1}^{n} c_i x^i$, $c_n = 1$. So

$$A_f = \begin{bmatrix} 0 & & & & -c_0 \\ 1 & 0 & & & -c_1 \\ & 1 & \ddots & & \vdots \\ & & & 0 & \vdots \\ & & & 1 & -c_{n-1} \end{bmatrix},$$

the **companion matrix** to f. The characteristic polnynomial is

$$c_T(x) = c_A(x) = c_{A_{f_1}}(x) \cdots c_{A_{f_r}}(x),$$

where

$$\begin{aligned} c_{A_f}(x) &= \begin{vmatrix} x & & & & c_0 \\ -1 & x & & & c_1 \\ & -1 & \ddots & & \vdots \\ & & & x & \vdots \\ & & & -1 & x + c_{n-1} \end{vmatrix} \\ &= x \begin{vmatrix} x & & & & c_1 \\ -1 & x & & & c_2 \\ & -1 & \ddots & & \vdots \\ & & & x & \vdots \\ & & & -1 & x + c_{n-1} \end{vmatrix} + (-1)^{n-1} c_0 \begin{vmatrix} -1 & x & & \\ & -1 & x & \\ & & \ddots & x \\ & & & -1 \end{vmatrix} \\ &= x \left(\frac{f - c_0}{x} \right) + c_0 \\ &= f. \end{aligned}$$

So $c_T(x) = f_1 \dots f_r$. Then $\mathrm{Ann}(V) = (f_r) = (m_T(x))$, where $m_T(x)$ is the **minimal polynomial**.

Assume $c_T(x)$ splits completely (e.g. F is algebraically closed. By the structure theorem, we can write

$$V \cong \bigoplus_{j=1}^{t} F[x]/(x - \lambda_j)^{n_j},$$

where $\lambda_j \in F$. Then

$$V = \bigoplus_{i=1}^{m} V_{\lambda_i}, \qquad \text{where } \bigoplus_{j=1}^{t_\lambda} F[x]/(x-\lambda_i)^{n_{\lambda,j}}$$

by grouping the terms with the same λ together. Let

$$V_{n,\lambda} = F[x]/(x-\lambda^n).$$

Take the basis $(x-\lambda)^{n-1}, (x-\lambda)^{n-2}, \ldots, 1$. Then

$$x \cdot (x - \lambda^{n-i} = \lambda(x-\lambda)^{n-i} + (x-\lambda)^{n-i+1}, \qquad 2 \le i \le n$$

$$x \cdot (x-\lambda)^{n-1} = \lambda(x-\lambda)^{n-1}$$

Then

$$J_{n,\lambda} \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix} \in M_n(F)$$

is called a **Jordan block**, and the matrix

$$A = \begin{bmatrix} J_{n_1,\lambda_1} & & \\ & \ddots & \\ & & J_{n_t,\lambda_t} \end{bmatrix}$$

represents $x : V \to V$ with respect to the basis

$$(x - \lambda_1^{n_1-1}, \ldots, 1, (x-\lambda_2)^{n_2-1}, \ldots, 1, \ldots, (x-\lambda_t)^{n_t-1}, \ldots, 1.$$

The characterisitc polynomial is

$$c_{A_{n,\lambda}}(x) = \begin{vmatrix} x-\lambda & -1 & & \\ & x-\lambda & & \\ & & \ddots & -1 \\ & & & x-\lambda \end{vmatrix} = (x-\lambda)^n.$$

1.2 Eigenvalues and eigenspaces

Proposition 1.1. *λ is an eigenvalue of T iff $\lambda = \lambda_i$ for some i (where λ_i are those appearing in the Jordan canonical form).*

Proof. Look at $J_{\lambda,n}$. Then $J_{\lambda,n}e_1 = \lambda e - 1$, and $(J_{\lambda,n} - \lambda I)e_i = e_{i-1}$. λ is an eigenvalue of R iff λ is on the diagonal of A. □

Definition 1.1. The **generalized eigenspace** of T for λ is

$$\{v \in V : (T - \lambda I)^m v = 0 \text{ for some } m \geq 0\}$$

Proposition 1.2. *$c_t(x)$ splits completely iff V s a direct sum of its generalized eigenspaces.*

Example 1.1. Let

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}.$$

The characteristic polynomial is $c_A(x) = (x-1)^3$. We have 3 possibilities for the Jordan canonical form:

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{bmatrix}.$$

Note that

$$A - I = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$$

has nullspace spanned by

$$\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$$

So we must be in the 2nd case. Look at

$$(A - I)e_1 = e_1 + e_2 - e_3.$$

Then we have the basis

$$B = (e_1, e_1 + e_2 + e_3, 2e_1 - e_2),$$

and A in this basis is

$$J = \begin{bmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{bmatrix} = Q^{-1}AQ,$$

where Q is the change of basis matrix from the standard basis to B. We can calculate

$$Q = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$